Progressions for the Common Core State Standards in Mathematics (draft)

3 December 2012

High School, Algebra

Overview

Two domains in middle school are important in preparing students for Algebra in high school. In the progression in The Number System, students learn to see all numbers as part of a unified system, and become fluent in finding and using the properties of operations to find the values of numerical expressions that include those numbers. The Expressions and Equations Progression describes how students extend their use of these properties to linear equations and expressions with letters.

The Algebra category in high school is very closely allied with the Functions category:

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the equation is in the form $y =$ (expression in x) or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent they define the same function.
- The solutions to an equation in one variable can be understood as the input values which yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing the functions defined by each side and finding the points where the graphs intersect.

Because of these connections, some curricula take a functions-based approach to teaching algebra, in which functions are introduced early and used as a unifying theme for algebra. Other more traditional approaches introduce functions later, after extensive work with expressions and equations. The separation between Algebra and Functions in the standards is not intended to indicate a preference between these two approaches. It is, however, intended to specify the difference as mathematical concepts between expressions and equations on the one hand and functions on the other. Students often enter college-level mathematics courses with an apparent confusion between all three of these concepts. For example, when asked to factor a quadratic expression a student might instead find the solutions of the corresponding quadratic equation. Or another student might attempt to simplify the expression $\frac{\sin x}{x}$ by cancelling the x's.

The Algebra standards are fertile ground for the standards for mathematical practice. Two in particular that stand out are MP7, Look for and make use of structure, and MP8, Look for and express regularity in repeated reasoning. Students are expected to see how the structure of an algebraic expression reveals properties of the function it defines. They are expected to move from repeated reasoning with the slope formula to writing equations in various forms for straight lines, rather than memorizing all those forms separately. In this way the Algebra standards provide focus in a way different from the K–8 standards. Rather than focusing on a few topics, students in high school focus on a few seed ideas that lead to many different techniques.

Seeing Structure in Expressions

Students have been seeing expressions since Kindergarten, starting with arithmetic expressions in Grades K–5 and moving on to algebraic expressions in Grades 6–8. The middle grades standards in Expression and Equations build a ramp from arithmetic in elementary school to more sophisticated work with algebraic expression in high school. As the complexity of expressions increase, students continue to see them as being built out of basic operations: they see expressions as sums of terms and products of factors.[A-SSE.1a]

For example, in the example on the right, students compare $P+Q$ and $2P$ by seeing $2P$ as $P+P$. They distinguish between $(Q-P)/2$ and $Q-P/2$ by seeing the first as a quotient where the numerators is a difference and the second as a difference where the second term is a quotient. This last example also illustrates how students are able to see complicated expressions as built up out of simpler ones.[A-SSE.1b] As another example, students can see the expression $5+(x-1)^2$ as a sum of a constant and a square; and then see that inside the square term is the expression $x-1$. The first way of seeing tells them that it is always greater than or equal to 5, since a square is always greater than or equal to 0; the second way of seeing tells them that the square term is zero when $x=1$. Putting these together they can see that this expression attains its minimum value, 5, when $x=1$. The margin lists other tasks from the Illustrative Mathematics project (illustrativemathematics.org) for

Animal populations

Suppose P and Q give the sizes of two different animal populations, where $Q > P$. In 1–4, which of the given pair of expressions is larger? Briefly explain your reasoning in terms of the two populations.

1. $P+Q$ and $2P$
2. $\dfrac{P}{P+Q}$ and $\dfrac{P+Q}{2}$
3. $(Q-P)/2$ and $Q-P/2$
4. $P+50t$ and $Q+50t$

A-SSE.1a Interpret expressions that represent a quantity in terms of its context.

a Interpret parts of an expression, such as terms, factors, and coefficients.

A-SSE.1b Interpret expressions that represent a quantity in terms of its context.

b Interpret complicated expressions by viewing one or more of their parts as a single entity.

A-SSE.1.

Initially, the repertoire of operations for building up expressions is limited to the operations of arithmetic: addition, subtraction, multiplication and division (with the addition in middle grades of exponent notation to represent repeated multiplication). By the time they get to college, students have expanded that repertoire to include functions such as the square root function, exponential functions, and trigonometric functions.

For example, students in physics classes might be expected see the expression

$$L_0\sqrt{1-\frac{v^2}{c^2}},$$

which arises in the theory of special relativity, as the product of the constant L_0 and a term that is 1 when $v = 0$ and 0 when $v = c$—and furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$.

Seeing structure in expressions entails a dynamic view of an algebraic expression, in which potential rearrangements and manipulations are ever present.[A-SSE.2] An important skill for college readiness is the ability to try out possible manipulations mentally without having to carry them out, and to see which ones might be fruitful and which not. For example, a student who can see

$$\frac{(2n+1)n(n+1)}{6}$$

as a polynomial in n with leading coefficient $\frac{1}{3}n^3$ has a leg up when it comes to calculus; a student who can mentally see the equivalence

$$\frac{R_1R_2}{R_1+R_2} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$$

without a laborious pencil and paper calculation is better equipped for a course in electrical engineering.

The standards avoid talking about simplification, because it is often not clear what the simplest form of an expression is, and even in cases where that is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as illustrated in the problem in the margin.[A-SSE.3]

For example, there are three commonly used forms for a quadratic expression:

- Standard form (e.g. $x^2 - 2x - 3$)
- Factored form (e.g. $(x+1)(x-3)$)

Illustrations of interpreting the structure of expression

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Delivery Trucks
- Kitchen Floor Tiles
- Increasing or Decreasing? Variation 1
- Mixing Candies
- Mixing Fertilizer
- Quadrupling Leads to Halving
- The Bank Account
- The Physics Professor
- Throwing Horseshoes
- Animal Populations
- Equivalent Expressions
- Sum of Even and Odd

A-SSE.2 Use the structure of an expression to identify ways to rewrite it.

Which form is "simpler"?

A container of ice cream is taken from the freezer and sits in a room for t minutes. Its temperature in degrees Fahrenheit is $a - b \cdot 2^{-t} + b$, where a and b are positive constants.
Write this expression in a form that shows that the temperature is always

1. Less than $a + b$
2. Greater than a

The form $a + b - b \cdot 2^{-t}$ for the temperature shows that it is $a + b$ minus a positive number, so always less than $a + b$. On the other hand, the form $a + b(1 - 2^{-t})$ reveals that the temperature is always greater than a, cause it is a plus a positive number.

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

- Vertex (or complete square) form (e.g. $(x-1)^2-4$).

Each is useful in different ways. The traditional emphasis on simplification as an automatic procedure might lead students to automatically convert the second two forms to the first, before considering which form is most useful in a given context.[A-SSE.3ab] This can lead to time consuming detours in algebraic work, such as solving $(x+1)(x-3)=0$ by first expanding and then applying the quadratic formula.

The introduction of rational exponents and systematic practice with the properties of exponents in high school widen the field of operations for manipulating expressions.[A-SSE.3c] For example, students in later algebra courses who study exponential functions see

$$P(1+\frac{r}{12})^{12n} \quad \text{as} \quad P\left((1+\frac{r}{12})^{12}\right)^n$$

in order to understand formulas for compound interest.

Much of the ability to see and use structure in transforming expressions comes from learning to recognize certain fundamental techniques. One such technique is recognizing internal cancellations, as in the expansion

$$(a-b)(a+b)=a^2-b^2.$$

An impressive example of this is

$$(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)=x^n-1,$$

in which all the terms cancel except the end terms. This identity is the foundation for the formula for the sum of a finite geometric series.[A-SSE.4]

a Factor a quadratic expression to reveal the zeros of the function it defines.

b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c Use the properties of exponents to transform expressions for exponential functions.

Illustrations of writing expressions in equivalent forms

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Ice Cream
- Increasing or Decreasing? Variation 2
- Profit of a company
- Seeing Dots

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Arithmetic with Polynomials and Rational Expressions

The development of polynomials and rational expressions in high school parallels the development of numbers in elementary school. In elementary school students might initially see expressions like $8+3$ and 11, or $\frac{3}{4}$ and 0.75, as fundamentally different: $8+3$ might be seen as describing a calculation and 11 is its answer; $\frac{3}{4}$ is a fraction and 0.75 is a decimal. Gradually they come to see numbers as forming a unified system, the number system, represented by points on the number line, and these different expressions are different ways of naming an underlying thing, a number.

A similar evolution takes place in algebra. At first algebraic expressions are simply numbers in which one or more letters are used to stand for a number which is either unspecified or unknown. Students learn to use the properties of operations to write expressions in different but equivalent forms. At some point they see equivalent expressions, particularly polynomial and rational expressions, as naming some underlying thing.[A-APR.1] There are at least two

A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

ways this can go. If the function concept is developed before or concurrently with the study of polynomials, then a polynomial can be identified with the function it defines. In this way $x^2 - 2x - 3$, $(x+1)(x-3)$, and $(x-1)^2 - 4$ are all the same polynomial because they all define the same function. Another approach is to think of polynomials as elements of a formal number system, in which you introduce the "number" x and see what numbers you can write down with it. Each approach has its advantages and disadvantages; the former approach is more common. Whichever is chosen, a curricular implementation might not necessarily explicitly state the choice, but should nonetheless be constructed in accordance with the implicit choice that has been made.

Either way, polynomials and rational expressions come to form a system in which things can be added, subtracted, multiplied and divided.[A-APR.7] Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers.

A-APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers.[A-APR.4] For example, students can explore the sequence of squares

A-APR.4 Prove polynomial identities and use them to describe numerical relationships.

$$1, 4, 9, 16, 25, 36, \ldots$$

and notice that the differences between them—3, 5, 7, 9, 11—are consecutive odd integers. This mystery is explained by the polynomial identity

$$(n+1)^2 - n^2 = 2n + 1.$$

A more complex identity,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2,$$

allows students to generate Pythagorean triples. For example, taking $x = 1$ and $y = 2$ in this identity yields $5^2 = 3^2 + 4^2$.

A particularly important polynomial identity, treated in advanced courses, is the Binomial Theorem[A-APR.5]

A-APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \binom{n}{3} x^{n-3} y^3 + \cdots + y^2,$$

for a positive integer n. The binomial coefficients can be obtained using Pascal's triangle

$n=0$:					1				
$n=1$:				1		1			
$n=2$:			1		2		1		
$n=3$:		1		3		3		1	
$n=4$:	1		4		6		4		1

in which each entry is the sum of the two above. Understanding why this rule follows algebraically from

$$(x+y)(x+y)^{n-1} = (x+y)^n$$

+ is excellent exercise in abstract reasoning (MP2) and in expressing
+ regularity in repeated reasoning (MP8).

Viewing polynomials as functions leads to explorations of a different nature. Polynomial functions are, on the one hand, very elementary, in that, unlike trigonometric and exponential functions, they are built up out of the basic operations of arithmetic. On the other hand, they turn out to be amazingly flexible, and can be used to approximate more advanced functions such as trigonometric and exponential functions. Although students only learn the complete story here if and when they study calculus, experience with constructing polynomial functions satisfying given conditions is useful preparation not only for calculus, but for understanding the mathematics behind curve-fitting methods used in applications to statistics and computer graphics.

A simple step in this direction is to construct polynomial functions with specified zeros.[A-APR.3] This is the first step in a progression which can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

A-APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

The analogy between polynomials and integers carries over to the idea of division with remainder. Just as in Grade 4 students find quotients and remainders of integers,[4.NBT.6] in high school they find quotients and remainders of polynomials.[A-APR.6] The method of polynomial long division is analogous to, and simpler than, the method of integer long division.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

A-APR.6 Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A particularly important application of polynomial division is the case where a polynomial $p(x)$ is divided by a linear factor of the form $x - a$, for a real number a. In this case the remainder is the value $p(a)$ of the polynomial at $x = a$.[A-APR.2] It is a pity to see this topic reduced to "synthetic division," which reduced the method to a matter of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a techique.

A-APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

A consequency of the Remainder Theorem is to establish the equivalence between linear factors and zeros that is the basis of much work with polynomials in high school: the fact that $p(a) = 0$ if and only if $x - a$ is a factor of $p(x)$. It is easy to see if $x - a$ is a factor then $p(a) = 0$. But the Remainder Theorem tells us that we can write

$$p(x) = (x - a)q(x) + p(a) \quad \text{for some polynomial } q(x).$$

In particular, if $p(a) = 0$ then $p(x) = (x - a)q(x)$, so $x - a$ is a factor of $p(x)$.

Creating Equations

Students have been writing equations, mostly linear equations, since middle grades. At first glance it might seem that the progression from middle grades to high school is fairly straightforward: the repertoire of functions that is acquired during high school allows students to create more complex equations, including equations arising from linear and quadratic functions, and simple rational and exponential functions;[A-CED.1] students are no longer limited largely to linear equations in modeling relationships between quantities with equations in two variables;[A-CED.2]; and students start to work with inequalities and systems of equations.[A-CED.3]

Two developments in high school complicate this picture. First, students in high school start using parameters in their equations, to represent whole classes of equations[F-LE.5] or to represent situations where the equation is to be adjusted to fit data.•

Second, modeling becomes a major objective in high school. Two of the standards just cited refer to "solving problems" and "interpreting solutions in a modeling context." And all the standards in the Creating Equations domain carry a modeling star, denoting their connection with the Modeling category in high school. This connotes not only an increase in the complexity of the equations studied, but an upgrade of the student's ability in every part of the modeling cycle, shown in the margin.

Variables, parameters, and constants Confusion about these terms plagues high school algebra. Here we try to set some rules for using them. These rules are not purely mathematical; indeed, from a strictly mathematical point of view there is no need for them at all. However, users of equations, by referring to letters as variables, parameters, or constants, can indicate how they intend to use the equations. This usage can be helpful if it is consistent.

In elementary and middle grades, life is easy. Elementary students solve problems with an unknown quantity, might use a symbol to stand for that quantity, and might call the symbol an unknown.[1.OA.2] In middle school students use variables systematically.[6.EE.6] They work with equations in one variable, such as $p + 0.05p = 10$ or equations in two variables such as $d = 5 + 5t$, relating two varying quantities.• In each case, apart from the variables, the numbers in the equation are given explicitly. The latter use presages the use of varibles to define functions.

In high school, things start to get complicated. For example, students consider the general equation for a straight line, $y = mx + b$. Here they are expected to understand that m and b are fixed for any given straight line, and that by varying m and b we obtain a whole family of straight lines. In this situation, m and b are called parameters. Of course, in an episode of mathematical work, the perspective could change; students might end up solving equations for m and b. Judging whether to explicitly indicate this—"now we

A-CED.1 Create equations and inequalities in one variable and use them to solve problems.

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

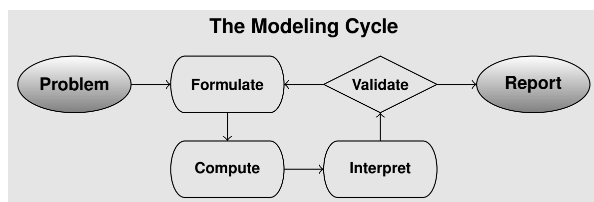
A-CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

•

> Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

CCSSM, page 73



1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

• Some writers prefer to retain the term unknown" for the first situation and the word "variable" for the second. This is not the usage adopted in the Standards.

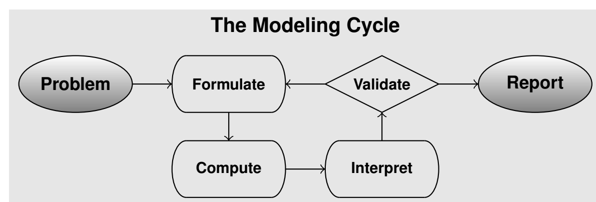
will regard the parameters as variables"—or whether to ignore it and just go ahead and solve for the parameters is a matter of pedagogical judgement.

Sometimes, and equation like $y = mx + b$ is used not to work with a parameterized family of equations but to consider the general form of an equation and prove something about it. For example, you might want take two points (x_1, y_1) and (x_2, y_2) on the graph of $y = mx + b$ and show that the slope between them is m. In this situation you might refer to m and b as constants rather than as parameters.

Finally, there are situations where an equation is used to describe the relationship between a number of different quantities, two which none of these terms apply.[A-CED.4] For example, Ohm's Law $V = IR$ relates the voltage, current, and resistance of an electrical circuit. An equation used in this way is sometimes called a formula. It is perhaps best to avoid entirely using the terms variable, parameter or constant when working with this formula, since there are 6 different ways it can be viewed as a defining one quantity as a function of the other with a third held constant.

A-CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Different curricular implementations of the standards might navigate these terminological shoals differently (including trying to avoid them entirely).



Modeling with equations Consider the *Formulate* node in the modeling cycle. In elementary school students learn to formulat an equation to solve a word problem. For example, in solving

> Selina bought a shirt on sale that was 20% less than the original price. The original price was \$5 more than the sale price. What was the original price? Explain or show work.

students might let p be the original price in dollars and then express the sale price in terms of p in two different ways and set them equal. On the one hand the sale price is 20% less than the original price, and so equal to $p - 0.2p$. On the other hand it is \$5 less than the original price, and so equal to $p - 5$. Thus they want to solve the equation

$$p - 0.2p = p - 5.$$

In this task, he formulation of the equation tracks the text of the problem fairly closely, but requires more than a keyword reading of the text. For example, the second sentence needs to be reinterpreted as "the sale price is \$5 less than the original price." Since the words "less" and "more" are typically the subject of schemes for guessing the operation required in a problem without reading it, this shift is significant, and prepares students to read more difficult and realistic task statements.

Indeed, in a typical high school modeling problem, there might be significantly different ways of going about a problem depending on the choices made, and students must be much more strategic in formulating the model.

The *Compute* node of the modeling cycle is dealt with in the next section, on solving equations.

The *Interpret* node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied.

The *Validate* node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself (MP4).

Reasoning with Equations and Inequalities

Equations in one variable

A naked equation, such as $x^2 = 4$, without any surrounding text, is merely a sentence fragment, neither true nor false, since it contains a variable x about which nothing is said. A written sequence of steps to solve an equation, such as in the margin, is code for a narrative line of reasoning using words like "if", "then", "for all" and "there exists." In the process of learning to solve equations, students learn certain standard "if-then" moves, for example "if $x = y$ then $x + 2 = y + 2$." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in the standards in this domain is that students understand that solving equations is a process of reasoning.[A-REI.1] This does not necessarily mean that they always write out the full text; part of the advantage of algebraic notation is its compactness. Once students know what the code stands for, they can start writing in code. Thus, eventually students might make $x^2 = 4 \implies x = \pm 2$ one step.[2]

Understanding solving equations as a process of reasoning demystifies "extraneous" solutions that can arise under certain solution procedures.[A-REI.2] The flow of reasoning is forward, from the assumption that a number x satisfies the equation to a list of possibilites for x. But not all the steps are necessarily reversible, and so it is not necessarily true that every number in the list satisfies the equation. For example, it is true that if $x = 2$ then $x^2 = 4$. But it is not true that if $x^2 = 4$ then $x = 2$ (it might be that $x = -2$). Squaring both sides of an equation is a typical example of an irreversible step; another is multiplying both sides of the equation by a quantity that might be zero (see margin for examples).

With an understanding of solving equations as a reasoning process, students can organize the various methods for solving different

[2]It should be noted, however, that calling this step "taking the square root of both sides" is dangerous, since it leads to the erroneous belief that $\sqrt{4} = \pm 2$.

Fragments of reasoning

$$
\begin{aligned}
x^2 &= 4 \\
x^2 - 4 &= 0 \\
(x-2)(x+2) &= 0 \\
x &= 2, -2
\end{aligned}
$$

This sequence of equations is short-hand for a line of reasoning: "If x is a number whose square is 4, then $x^2 - 4 = 0$. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers x, it follows that $(x-2)(x+2) = 0$. So either $x - 2 = 0$, in which case $x = 2$, or $x + 2 = 0$, in which case $x = -2$." More might be said: a justification of the last step, for example, or a check that 2 and -2 actually do satisfy the equation, which has not been proved by this line of reasoning.

A-REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

types of equations into a coherent picture. For example, solving linear equations involves only steps that are reversible (adding a constant to both sides, multiplying both sides by a non-zero constant, transforming an expression on one side into an equivalent expression). Therefore solving linear equations does not produce extraneous solutions.[A-REI.3] The process of completing the square also involves only this same list of steps, and so converts any quadratic equation into an equivalent equation of the form $(x-p)^2 = q$ that has exactly the same solutions[A-REI.4a] The latter equation is easy to solve by the reasoning explained above.

This example sets up a theme that reoccurs throughout algebra; finding ways of transforming equations into certain standard forms that have the same solutions. For example, any exponential equation can be transformed into the form $b^x = a$, the solution to which is (by definition) a logarithm.

It is traditional for students to spend a lot of time on various techniques of solving quadratic equations, which are often presented as if they are completely unrelated (factoring, completing the square, the quadratic formula). In fact, as we have seen, the key step in completing the square, going from $x^2 = q$ to $x = \pm\sqrt{q}$, involves at its heart factoring. And the quadratic formula is nothing more than an encapsulation of the method of completing the square. Rather than long drills on techniques of dubious value, students with an understanding of the underlying reasoning behind all these methods are opportunistic in their application, choosing the method that bests suits the situation at hand.[A-REI.4b]

A-REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.4a Solve quadratic equations in one variable.

a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Systems of equations

Student work with solving systems of equations starts the same way as work with solving equations in one variable; with an understanding of the reasoning behind the various techniques.[A-REI.5] An important step is realizing that a solution to a system of equations must be a solution all of the equations in the system simultaneously. Then the process of adding one equation to another is understood as "if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum of the left sides of the two equations is equal to the sum of the right sides." Since this reasoning applies equally to subtraction, the process of adding one equation to another is reversible, and therefore leads to an equivalent system of equations.

A-REI.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Understanding these points for the particular case of two equations in two variables is preparation for more general situations. Such systems also have the advantage that a good graphical visualization is available; a pair (x, y) satisfies two equations in two variables if it is on both their graphs, and therefore an intersection point of the graphs.[A-REI.6]

A-REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Another important method of solving systems is the method of substitution. Again this can be understood in terms of simultaneity;

if (x, y) satisfies two equations simultaneously, then the expression for y in terms of x obtained from the first equation should form a true statement when substituted into the second equation. Since a linear equation can always be solved for one of the variables in it, this is a good method when just one of the equations in a system is linear. [A-REI.7]

In more advanced courses, students see systems of linear equations in many variables as single matrix equations in vector variables.[A-REI.8, A-REI.9]

A-REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

A-REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Visualizing solutions graphically

Just as the algebraic work with equations can be reduced to a series of algebraic moves unsupported by reasoning, so can the graphical visualization of solutions. The simple idea that an equation $f(x) = g(x)$ can be solved (approximately) by graphing $y = f(x)$ and $y = g(x)$ and finding the intersection points involves a number of pieces of conceptual understanding.[A-REI.11] This seemingly simple method, often treated as obvious, involves the rather sophisticated move of reversing the reduction of an equation in two variables to an equation in one variable. Rather, it seeks to convert an equation in one variable, $f(x) = g(x)$, to a system of equations in two variables, $y = f(x)$ and $y = g(x)$, by introducing a second variable y and setting it equal to each side of the equation. If x is a solution to the original equation then $f(x)$ and $g(x)$ are equal, and thus (x, y) is a solution to the new system. This reasoning is often tremendously compressed and presented as obvious graphically; in fact following it graphically in a specific example can be instructive. [Give example in margin.]

A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Fundamental to all of this is a simple understanding of what a graph of an equation in two variables means.[A-REI.10]

A-REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).